NON-ORIENTABLE LAGRANGIAN COBORDISMS BETWEEN LEGENDRIAN KNOTS

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Abstract. In the symplectization of standard contact 3-space, $\mathbb{R}\times\mathbb{R}^3$, it is known that an orientable Lagrangian cobordism between a Legendrian knot and itself, also known as an orientable Lagrangian endocobordism for the Legendrian knot, must have genus 0. We show that any Legendrian knot has a non-orientable Lagrangian endocobordism, and that the crosscap genus of such a non-orientable Lagrangian endocobordism must be a positive multiple of 4. The more restrictive exact, non-orientable Lagrangian endocobordisms do not exist for any exactly fillable Legendrian knot but do exist for any stabilized Legendrian knot. Moreover, the relation defined by exact, non-orientable Lagrangian cobordism on the set of stabilized Legendrian knots is symmetric and defines an equivalence relation, a contrast to the non-symmetric relation defined by orientable Lagrangian cobordisms.

1. Introduction

Smooth cobordisms are a common object of study in topology. Motivated by ideas in symplectic field theory, [19], Lagrangian cobordisms that are cylindrical over Legendrian submanifolds outside a compact set have been an active area of research interest. Throughout this paper, we will study Lagrangian cobordisms in the symplectization of the standard contact $\mathbb{R}^3$, namely the symplectic manifold $(\mathbb{R}\times\mathbb{R}^3, d(e^t\alpha))$ where $\alpha = dz - y dx$, that coincide with the cylinders $\mathbb{R}\times\Lambda_+$ (respectively, $\mathbb{R}\times\Lambda_-$) when the $\mathbb{R}$-coordinate is sufficiently positive (respectively, negative). Our focus will be on non-orientable Lagrangian cobordisms between Legendrian knots Λ_+ and Λ_- and non-orientable Lagrangian endocobordisms, which are non-orientable Lagrangian cobordisms with $\Lambda_+ = \Lambda_-$.

Smooth endocobordisms in $\mathbb{R}\times\mathbb{R}^3$ without the Lagrangian condition are abundant: for any smooth knot $K \subset \mathbb{R}^3$, and an arbitrary $j \geq 0$, there is a smooth 2-dimensional orientable submanifold M of genus j so that M agrees with the cylinder $\mathbb{R}\times K$ when the $\mathbb{R}$ coordinate lies outside an interval $[T_-, T_+]$; the analogous statement holds for non-orientable M and crosscap genus[1] when $j > 0$. For any Legendrian knot Λ, it is easy to construct an orientable Lagrangian endocobordism of genus 0, namely the trivial Lagrangian cylinder $\mathbb{R}\times\Lambda$. In fact, with the added Lagrangian condition, *orientable* Lagrangian endocobordisms must be concordances:

[1] the number of real projective planes in a connected sum decomposition

Theorem (Chantraine, [8])**.** *For any Legendrian knot Λ, any orientable, Lagrangian endocobordism for Λ must have genus 0.*

Non-orientable Lagrangian endocobordisms also exist and have topological restrictions:

Theorem 1.1. *For an arbitrary Legendrian knot Λ, there exists a non-orientable Lagrangian endocobordism for Λ of crosscap genus g if and only if $g \in 4\mathbb{Z}^+$.*

Theorem 1.1 is proved in Theorem 3.2 and Theorem 3.3. The fact that the crosscap genus of a non-orientable Lagrangian endocobordism must be a positive multiple of 4 follows from a result of Audin about the obstruction to the Euler characteristic of closed, Lagrangian submanifolds in $\mathbb{R}^4$, [1]. It is easy to construct *immersed* Lagrangian endocobordisms; the existence of the desired embedded endocobordisms follows from Lagrangian surgery, as developed, for example, by Polterovich in [35].

Of special interest are Lagrangian cobordisms that satisfy an additional "exactness" condition. Exactness is known to be quite restrictive: by a foundational result of Gromov, [28], there are no closed, exact Lagrangian submanifolds in $\mathbb{R}^{2n}$ with its standard symplectic structure. The non-closed trivial Lagrangian cylinder $\mathbb{R} \times \Lambda$ is exact, and Section 2 describes some general methods to construct exact Lagrangian cobordisms. In contrast to Theorem 1.1, there are some Legendrians that do not admit *exact*, non-orientable Lagrangian endocobordisms:

Theorem 1.2. *There does not exist an exact, non-orientable Lagrangian endocobordism for any Legendrian knot Λ that is exactly orientably or non-orientably fillable.*

A Legendrian knot Λ is exactly fillable if there exists an exact Lagrangian cobordism that is cylindrical over Λ at the positive end and does not intersect $\{T_-\} \times \mathbb{R}^3$, for $T_- \ll 0$; precise definitions can be found in Section 2. Theorem 1.2 is proved in Section 4; it follows from the Seidel Isomorphism, which relates the topology of a filling to the linearized contact cohomology of the Legendrian at the positive end. Theorem 1.2 implies that on the set of Legendrian knots in $\mathbb{R}^3$ that are exactly fillable, orientably or not, the relation defined by exact, non-orientable Lagrangian cobordism is *anti-reflexive* and *anti-symmetric*, see Corollary 4.2. Figure 6 gives some particular examples of Legendrians that are exactly fillable and thus do not admit exact, non-orientable Lagrangian endocobordisms. Many of these examples are maximal *tb* Legendrian representatives of twist and torus knots. In fact, using the classification results of Etnyre and Honda, [23], and Etnyre, Ng, and Vértesi, [24], we show:

Corollary 1.3. *Let K be the smooth knot type of either a twist knot or a positive torus knot or a negative torus knot of the form $T(-p, 2k)$, for p odd and $p > 2k > 0$. Then any maximal tb Legendrian representative of K does not have an exact, non-orientable Lagrangian endocobordism.*

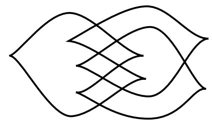

FIGURE 1. Does the max *tb* Legendrian representative of $m(8_{19})$ have an exact, non-orientable Lagrangian endocobordism?

However, stabilized Legendrian knots do admit exact, non-orientable Lagrangian endocobordisms: a Legendrian knot is said to be stabilized if, after Legendrian isotopy, a strand contains a zig-zag as shown in Figure 4.

Theorem 1.4. *For any stabilized Legendrian knot Λ and any $k \in \mathbb{Z}^+$, there exists an exact, non-orientable Lagrangian endocobordism for Λ of crosscap genus $4k$.*

Some Legendrian knots are neither exactly fillable nor stabilized. Thus, a natural quetion is:

Question 1.5. *If a Legendrian knot is not exactly fillable and is not stabilized, does it have an exact, non-orientable Lagrangian endocobordism? In particular, does the Legendrian representative of $m(8_{19}) = T(-4,3)$ with maximal tb shown in Figure 1 have an exact, non-orientable Lagrangian endocobordism?*

The max *tb* version of $m(8_{19})$ is not exactly fillable since the upper bound on the *tb* invariant for all Legendrian representatives of $m(8_{19})$ given by the Kauffman polynomial is not sharp; Section 6 for more details and related questions.

Given the existence of exact, non-orientable Lagrangian endocobordisms for a stabilized Legendrian, it is natural to ask: What Legendrian knots can appear as a "slice" of such an endocobordism? The parallel question for orientable Lagrangian endocobordisms has been studied in [9, 4, 12]. The non-orientable version of this question is closely tied to the question of whether non-orientable Lagrangian cobordisms define an equivalence relation on the set of Legendrian knots. By a result of Chantraine, [8], it is known that the relation defined on the set of Legendrian knots by *orientable* Lagrangian cobordism is *not* an equivalence relation since symmetry fails. In fact, the relation defined on the set of *stabilized* Legendrian knots by exact, non-orientable Lagrangian cobordism is symmetric: see Theorem 5.2. It is then easy to deduce:

Theorem 1.6. *On the set of stabilized Legendrian knots, the relation defined by exact, non-orientable Lagrangian cobordism is an equivalence relation. Moreover, all stabilized Legendrian knots are equivalent with respect to this relation.*

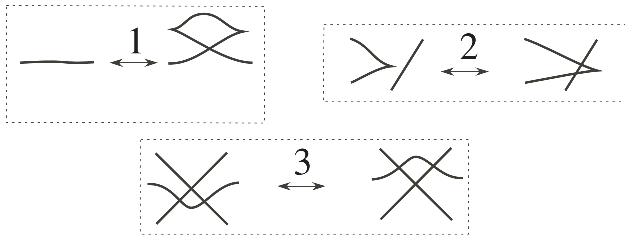

FIGURE 2. The three Legendrian Reidemeister moves.

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2. Background

In this section, we give some basic background on Legendrian and Lagrangian submanifolds.

2.1. **Contact Manifolds and Legendrian Submanifolds.** Below is some basic background on contact manifolds and Legendrian knots. More information can be found, for example, in [21] and [22].

A **contact manifold** (Y, ξ) is an odd-dimensional manifold together with a contact structure, which consists of a field of maximally non-integrable tangent hyperplanes. The **standard contact structure** on $\mathbb{R}^3$ is the field $\xi_p = \ker \alpha_0(p)$, for $\alpha_0(x, y, z) = dz - ydx$. A **Legendrian link** is a submanifold, Λ, of $\mathbb{R}^3$ diffeomorphic to a disjoint union of circles so that for all $p \in \Lambda$, $T_p\Lambda \subset \xi_p$; if, in addition, Λ is connected, Λ is a **Legendrian knot**. It is common to examine Legendrian links from their xz-projections, known as their **front projections**. A Legendrian link will generically have an immersed front projection with semi-cubical cusps and no vertical tangents; any such projection can be uniquely lifted to a Legendrian link using $y = dz/dx$.

Two Legendrian links Λ_0 and Λ_1 are **equivalent Legendrian links** if there exists a 1-parameter family of Legendrian links Λ_t joining Λ_0 and Λ_1. In fact, Legendrian links Λ_0, Λ_1 are equivalent if and only if their front projections are equivalent by planar isotopies that do not introduce vertical tangents and the **Legendrian Reidemeister moves** as shown in Figure 2.

Every Legendrian knot has a Legendrian representative. In fact, every Legendrian knot has an infinite number of different Legendrian representatives. For example, Figure 3 shows three different oriented Legendrians that are all topologically the unknot. These unknots can be distinguished

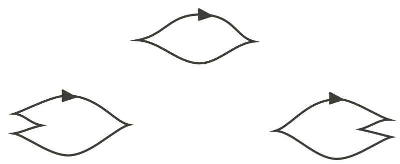

FIGURE 3. Three different Legendrian unknots; the one with maximal tb invariant of -1 and two others obtained by $\pm$-stabilizations.

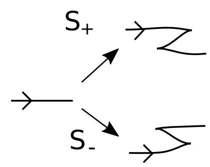

FIGURE 4. The positive (negative) stabilization of an oriented knot is obtained by introducing a down (an up) zig-zag.

by classical Legendrian invariant numbers, the Thurston-Bennequin, tb, and rotation, r. These invariants can easily be computed from a front projection; see, for example, [5].

The two unknots in the second line of Figure 3 are obtained from the one at the top by stabilization. In general, from an oriented Legendrian Λ, one can obtain oriented Legendrians $S_\pm(\Lambda)$: the **positive (negative) stabilization,** S_+ **(**S_-**),** is obtained by replacing a portion of a strand with a strand that contains a down (up) **zig-zag**, as shown in Figure 4. This stabilization procedure will not change the underlying smooth knot type but will decrease the Thurston-Bennequin number by 1; adding an up (down) zig-zag will decrease (increase) the rotation number by 1. It is possible to move a zig-zag to any strand of a Legendrian knot, [26]. For any smooth knot type, all Legendrian representatives can be represented by a mountain range that records the possible tb and r values; many examples of known and conjectured mountain ranges can be found in the Legendrian knot atlas of Chongchitmate and Ng, [11].

2.2. **Symplectic Manifolds, Lagrangian Submanifolds, and Lagrangian Cobordisms.** We will now discuss some basic concepts in symplectic geometry. Additional background can be found, for example, in [32].

A **symplectic manifold** (M, ω) is an even-dimensional manifold together with a 2-form ω that is closed and non-degenerate; when ω is an exact 2-form, $(M, \omega = d\beta)$ is said to be an **exact symplectic manifold**. A basic example of an exact symplectic manifold is $(\mathbb{R}^4, \omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$. The cobordisms constructed in this paper live inside the symplectic manifold that is constructed as the symplectization of $(\mathbb{R}^3, \xi_0 = \ker \alpha_0)$, namely, $\mathbb{R} \times \mathbb{R}^3$

with symplectic form given by $\omega = d(e^t\alpha_0)$. In fact, the symplectization $(\mathbb{R}\times\mathbb{R}^3,\omega)$ is exactly symplectically equivalent to the standard $(\mathbb{R}^4,\omega_0)$, see for example [6].

A **Lagrangian submanifold** L of a 4-dimensional symplectic manifold (M,ω) is a 2-dimensional submanifold so that $\omega|_L = 0$. When M is an exact symplectic manifold, $\omega = d\beta$, $\beta|_L$ is necessarily a closed 1-form; when, in addition, $\beta|_L$ is an exact 1-form, $\beta|_L = df$, then L is said to be an **exact Lagrangian submanifold**.

Remark 2.1. There is a (non-exact) Lagrangian torus in the standard symplectic $\mathbb{R}^4$: this can be seen as the product of two embedded circles in each of the (x_1, y_1) and (x_2, y_2) planes. By classical algebraic topology, it follows that the torus is the only compact, orientable surface that admits a Lagrangian embedding into $\mathbb{R}^4$, [3].

We will focus on non-compact Lagrangians that are cylindrical over Legendrians.

Definition 2.1. Let Λ_-, Λ_+ be Legendrian links in $\mathbb{R}^3$.

(1) A Lagrangian submanifold without boundary $L \subset \mathbb{R}\times\mathbb{R}^3$ is a **Lagrangian cobordism from Λ_+ to Λ_-** if it is of the form

$$L = ((-\infty, T_-] \times \Lambda_-) \cup \overline{L} \cup ([T_+, +\infty) \times \Lambda_+),$$

for some $T_- < T_+$, where $\overline{L} \subset [T_-, T_+] \times \mathbb{R}^3$ is compact with boundary $\partial\overline{L} = (\{T_-\} \times \Lambda_-) \cup (\{T_+\} \times \Lambda_+)$.

(2) A Lagrangian cobordism from Λ_+ to Λ_- is **orientable (resp., non-orientable)** if L is orientable (resp., non-orientable).

(3) A Lagrangian cobordism from Λ_+ to Λ_- is **exact** if L is exact, namely $e^t\alpha_0|_L = df|_L$, and the primitive, f, is constant on the cylindrical ends: there exists constants $C_\pm$ so that

$$f|_{L\cap((-\infty,T_-)\times\mathbb{R}^3)} = C_-, \qquad f|_{L\cap((T_+,+\infty)\times\mathbb{R}^3)} = C_+.$$

A Legendrian knot Λ is **(exactly) fillable** if there exists an (exact) Lagrangian cobordism from $\Lambda_+ = \Lambda$ to $\Lambda_- = \emptyset$.

An important property of Lagrangian cobordisms is that they can be stacked/composed:

Lemma 2.2 (Stacking Cobordisms, [17])**.** *If L_{12} is an exact Lagrangian cobordism from $\Lambda_+ = \Lambda_1$ to $\Lambda_- = \Lambda_2$, and L_{23} is an exact Lagrangian cobordism from $\Lambda_+ = \Lambda_2$ to $\Lambda_- = \Lambda_3$, then there exists an exact Lagrangian cobordism L_{13} from $\Lambda_+ = \Lambda_1$ to $\Lambda_- = \Lambda_3$.*

Constructions of exact Lagrangian cobordisms are an active area of research. In this paper, we will use the fact that there exist exact Lagrangian cobordisms between Legendrians related by isotopy and surgery. The existence of exact Lagrangian cobordisms from isotopy is well-known, see, for example, [20], [8], [17], and [6].

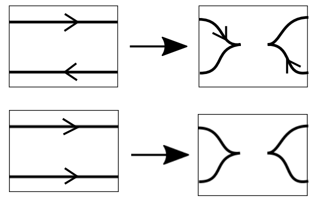

FIGURE 5. Orientable and Non-Orientable Legendrian surgeries.

Lemma 2.3 (Exact Cobordisms from Isotopy)**.** *Suppose that Λ and Λ' are isotopic Legendrian knots. Then there exists an exact, orientable Lagrangian cobordism from $\Lambda_+ = \Lambda$ to $\Lambda_- = \Lambda'$.*

Remark 2.2. In general, the trace of a Legendrian isotopy is not a Lagrangian cobordism. However it is possible to add a "correction term" so that it will be Lagrangian. More precisely, let $\lambda_t(u) = (x(t,u), y(t,u), z(t,u))$, $t \in \mathbb{R}$, be a Legendrian isotopy so that $\frac{\partial \lambda}{\partial t}(t,u)$ has compact support with $\operatorname{Im} \lambda_t(u) = \Lambda_-$ for $t \leq -T$ and $\operatorname{Im} \lambda_t(u) = \Lambda_+$ for $t \geq T$, and let

$$\eta(t,u) = \alpha_0 \left(\frac{\partial \lambda}{\partial t}(t,u) \right).$$

Then $\Gamma(t,u) = (t, x(t,u), y(t,u), z(t,u) + \eta(t,u))$ is an exact Lagrangian immersion. If $\eta(t,u)$ is sufficiently small, which can be guaranteed by making T sufficiently large, then $\Gamma(t,u)$ is an exact Lagrangian embedding.

In addition, Legendrians Λ and Λ' that differ by "surgery" can be connected by an exact Lagrangian cobordism. The 0-surgery operation can be viewed as a "tangle surgery": the replacement of a Legendrian 0-tangle, consisting of two strands with no crossings and no cusps, with a Legendrian ∞-tangle, consisting of two strands that each have 1 cusp and no crossings; see Figure 5. When the strands of the 0-tangle are oppositely oriented, this is an **orientable surgery**; otherwise this is a **non-orientable surgery**. In addition, by an index 1 surgery, it is known that the maximal *tb* Legendrian representative of the unknot, shown at the top of Figure 3, can be filled.

Lemma 2.4 (Exact Cobordisms from Surgery, [17, 14, 6])**.**

1. *Suppose that Λ_+ and Λ_- are Legendrian knots where Λ_- is obtained from Λ_+ by orientable (non-orientable) surgery, as shown in Figure 5. Then there exists an exact, orientable (non-orientable) Lagrangian cobordism from Λ_+ to Λ_-.*
2. *Suppose Λ_+ is the Legendrian unknot with tb equal to the maximum value of -1. Then there exists an exact, orientable Lagrangian filling of Λ_+.*

Remark 2.3. By Lemmas 2.3 and 2.4, to show there exists an exact Lagrangian cobordism from Λ_+ to Λ_-, it suffices to show that there is a string of Legendrian links $(\Lambda_+ = \Lambda_0, \Lambda_1, \ldots, \Lambda_n = \Lambda_-)$, where each Λ_{i+1} is obtained from Λ_i by a single surgery, as shown in Figure 5, and Legendrian

isotopy. In the case where each surgery is orientable, the exact Lagrangian cobordism will be orientable; in this case, the length n of this string must be even and will agree with twice the genus of the Lagrangian cobordism; for more details, see [5]. If there is at least one non-orientable surgery, the exact Lagrangian cobordism will be non-orientable and the length of the string agrees with the crosscap genus of the Lagrangian cobordism. To construct an exact Lagrangian filling of Λ_+, it suffices to construct such a string to $\Lambda_- = U$, where U is a trivial link of maximal *tb* Legendrian unknots.

3. Constructions of Non-orientable Lagrangian Endocobordisms

In this section, we show that *any* Legendrian knot has a non-orientable Lagrangian endocobordism with crosscap genus an arbitrary multiple of 4. We then show that it is not possible to get any other crosscap genera.

The strategy to show existence is to first construct an immersed orientable Lagrangian cobordism, and then apply "Lagrangian surgery" to modify it so that it is embedded. The following description of Lagrangian surgery follows Polterovich's construction, [35]; see also work of Lalonde and Sikorav, [30].

To state Lagrangian surgery precisely, we first need to explain the "index" of a double point. Suppose that x is a point of self-intersection of a generic, immersed, oriented 2-dimensional submanifold L of $\mathbb{R}^4$. Then $ind(x) \in \{\pm 1\}$ will denote the **index of self-intersection of** L at x: let (v_1, v_2) and (w_1, w_2) be positively oriented bases of the transverse tangent spaces at x, then

$$\text{ind}(x) = +1 \iff (v_1, v_2, w_1, w_2) \text{ is a positively oriented basis of } \mathbb{R}^4,$$

and otherwise $\text{ind}(x) = -1$.

By constructing a Lagrangian handle in a Darboux chart, it is possible to remove double points of a Lagrangian:

Lemma 3.1 (Lagrangian Surgery, [35])**.** *Let Σ be a 2-dimensional manifold. Suppose $\phi : \Sigma \to \mathbb{R}^4$ is a Lagrangian immersion, and $U \subset \mathbb{R}^4$ contains a single transversal double point x of ϕ. Then there exists a 2-dimensional manifold Σ', which is obtained by a Morse surgery on Σ, and a Lagrangian immersion $\phi' : \Sigma' \to \mathbb{R}^4$ so that*

1. $\text{Im}\,\phi = \text{Im}\,\phi'$ *on* $\mathbb{R}^4 - U$*;*
2. ϕ' *has no double points in* U.

Furthermore, let $\phi^{-1}(\{x\}) = \{p_1, p_2\} \subset \Sigma$. Then

1. *if p_1, p_2 are in disjoint components of Σ, then Σ' is obtained from Σ by a connect sum operation;*
2. *if p_1, p_2 are in the same component of Σ then:*
   (a) *if Σ is not oriented, $\Sigma' = \Sigma \# K$,*
   (b) *if Σ is oriented, then $\Sigma' = \Sigma \# T$, when* $\text{ind}(x) = +1$, *and $\Sigma' = \Sigma \# K$, when* $\text{ind}(x) = -1$,

   where K denotes the Klein bottle, and T denotes the torus.

We now have the necessary background to show the existence of a non-orientable Lagrangian endocobordism for any Legendrian knot:

Theorem 3.2. *For any Legendrian knot Λ and any $k \in \mathbb{Z}^+$, there exists a non-orientable Lagrangian endocobordism for Λ of crosscap genus $4k$.*

Proof. For an arbitrary Legendrian knot Λ, begin with cylindrical Lagrangian cobordism, $L = \mathbb{R} \times \Lambda \subset \mathbb{R} \times \mathbb{R}^3$, which is a space that is symplectically equivalent to the standard $\mathbb{R}^4$. As explained in Remark 2.1, there exists an embedded Lagrangian torus, T, so that $T \cap L = \emptyset$. After a suitable shift and perturbation, we can assume that L and T intersect at exactly two points, x_1 and x_2 where $\mathrm{ind}(x_1) = +1$ and $\mathrm{ind}(x_2) = -1$. By Lemma 2.4, Lagrangian surgery at x_1 results in the connected, oriented, immersed Lagrangian diffeomorphic to $(\mathbb{R} \times S^1)\#T$ with a double point at x_2 of index -1; a second Lagrangian surgery at x_2 results in a embedded, non-orientable Lagrangian cobordism diffeomorphic to $\mathbb{R} \times S^1 \times T \times K$, and thus of crosscap genus 4. Stacking these endocobordisms, using Lemma 2.2, produces an embedded, non-orientable Lagrangian cobordism of crosscap genus $4k$, for any $k \in \mathbb{Z}^+$. □

In fact, the possible crosscap genera that appeared in Theorem 3.2 are all that can exist:

Theorem 3.3. *Any non-orientable Lagrangian endocobordism in $\mathbb{R} \times \mathbb{R}^3$ must have crosscap genus $4k$, for some $k \in \mathbb{Z}^+$.*

This crosscap genus restriction is closely tied to Euler characteristic obstructions for *compact*, non-orientable submanifolds that admit Lagrangian embeddings in $(\mathbb{R}^4, \omega_0)$, or equivalently in $(\mathbb{R} \times \mathbb{R}^3, d(e^t\alpha))$:

Lemma 3.4 (Audin, [1])**.** *Any compact, non-orientable Lagrangian submanifold of $\mathbb{R} \times \mathbb{R}^3$ has an Euler characteristic divisible by 4.*

This result can be seen as an extension of a formula of Whitney that relates the number of double points of a smooth immersion to the Euler characteristic of the normal bundle of the immersion and thus of the tangent bundle of a Lagrangian immersion; see [1, 3].

Remark 3.1. Lemma 3.4 implies that any compact, non-orientable, Lagrangian submanifold L in $\mathbb{R} \times \mathbb{R}^3$ has crosscap genus $2+4j$, for some $j \geq 0$. There are explicit constructions of compact, non-orientable Lagrangian submanifolds of crosscap genus $2+4j$, for all $j > 0$, [27, 2]. It has been shown that there is no embedded, Lagrangian Klein bottle ($j = 0$), [33, 39].

To utilize the crosscap genus restrictions for compact Lagrangians, we will employ the following lemma, which shows that for any Lagrangian endocobordism, it is possible to construct a compact, non-orientable Lagrangian submanifold into which we can glue the compact portion of a Lagrangian endocobordism.

Lemma 3.5. *For any Legendrian knot $\Lambda \subset \mathbb{R}^3$, any open set $D \subset \mathbb{R}^3$ containing Λ, and any $T \in \mathbb{R}^+$, there exists a compact, non-orientable Lagrangian submanifold L in $\mathbb{R} \times \mathbb{R}^3$ so that*

$$L \cap ([-T, T] \times D) = [-T, T] \times \Lambda.$$

Proof. The strategy will be to construct a Lagrangian immersion of the torus, thought of as two finite cylinders with top and bottom circles identified, and then apply Lagrangian surgery to remove the immersion points. As a first step, we construct (non-disjoint) Lagrangian embeddings of two cylinders via Legendrian isotopies, Lemma 2.3. Namely, start with two disjoint copies of Λ: Λ in D and a translated version $\Lambda' \in \mathbb{R}^3 - D$. Now, for $t \in [0, t_2]$, consider Legendrian isotopies Λ_t of Λ and Λ'_t of Λ' that satisfy the following conditions: $\Lambda_t = \Lambda$, for all $t \in [0, t_2]$; $\Lambda'_t = \Lambda'$, for $t \in [0, t_1]$, and then for $t \in [t_1, t_2]$, Λ'_t is a Legendrian isotopy of Λ' so that $\Lambda'_{t_2} = \Lambda = \Lambda_{t_2}$. By repeating an analogous procedure for $t \in [-t_2, 0]$, we can obtain a smooth, immersion of the torus into $[-t_2, t_2] \times \mathbb{R}^3$. The arguments used to prove Lemma 2.3 (see Remark 2.2) show that for sufficiently large t_2, the image of the trace of these isotopies can be perturbed to two non-disjoint embedded Lagrangian cylinders that do not have any intersection points in $[-t_1, t_1] \times \mathbb{R}^3$. Then by applying Lagrangian surgery, Lemma 2.4, at each double point we get a compact, non-orientable Lagrangian submanifold L in $\mathbb{R} \times \mathbb{R}^3$ with the desired properties. $\square$

We are now ready to prove the crosscap genus restriction for arbitrary non-orientable, Lagrangian endocobordisms:

Proof of Theorem 3.3. Let C be a non-orientable Langrangian endocobordism. Suppose $C \subset \mathbb{R} \times D$ and C agrees with standard cylinder outside $[-T, T] \times \mathbb{R}^3$. By Lemma 3.5, there is a compact, non-orientable Lagrangian submanifold L in $\mathbb{R} \times \mathbb{R}^3$ so that

$$L \cap ([-T, T] \times D) = [-T, T] \times \Lambda.$$

Let L' be the Lagrangian submanifold obtained by removing the standard cylindrical portion of L in $[-T, T] \times D$ and replacing it with $C \cap ([-T, T] \times \mathbb{R}^3)$. Then L' will be a compact, non-orientable Lagrangian submanifold whose crosscap genus, $k(L')$, differs from the crosscap genus of L, $k(L)$, by the crosscap genus of C, $k(C)$: $k(L') = k(L) + k(C)$. By Lemma 3.4, there exist $j, j' \in \mathbb{Z}^+$ so that $k(L) = 2 + 4j$ and $k(L') = 2 + 4j'$. Thus we find that the crosscap genus of C, $k(C)$, must be divisible by 4. $\square$

Remark 3.2. For exact Lagrangian cobordisms that are constructed from isotopy and surgery, Lemmas 2.3 and 2.4, it is possible to show that the crosscap genus must be a multiple of 4 by an alternate argument that relies on a careful analysis of the possible changes to $tb(\Lambda)$ under surgery; [7].

4. Obstructions to Exact Non-Orientable Lagrangian Endocobordisms

We will now begin to focus on *exact*, non-orientable Lagrangian cobordisms. In this section, we will prove Theorem 1.2, which states that any Legendrian knot that is exactly fillable does not have an exact non-orientable Lagrangian endocobordism. The proof of this theorem will involve applying the Seidel Isomorphism, which relates the the topology of a filling to the linearized Legendrian contact cohomology of the Legendrian at the positive end. We will then apply Theorem 1.2 and give examples of maximal *tb* Legendrian knots that do not have exact, non-orientable Lagrangian endocobordisms.

We begin with a brief description of Legendrian contact homology; additonal background information can be found, for example, in [22]. Legendrian contact homology is a Floer-type invariant of a Legendrian submanifold that lies within Eliashberg, Givental, and Hofer's Symplectic Field Theory framework; [18, 19, 10]. It is possible to associate to a Legendrian submanifold $\Lambda \subset \mathbb{R}^3$ the stable, tame isomorphism class of an associative differential graded algebra (DGA), $(\mathcal{A}(\Lambda), \partial)$. The algebra is freely generated by the Reeb chords of Λ, and is graded using a Maslov index. The differential comes from counting pseudo-holomorphic curves in the symplectization of $\mathbb{R}^3$; for our interests, we will always use $\mathbb{Z}/2$ coefficients. **Legendrian contact homology**, namely the homology of $(\mathcal{A}(\Lambda), \partial)$, is a Legendrian invariant of Λ.

In general, it is difficult to extract information directly from the Legendrian contact homology. An important computational technique arises from the existence of augmentations of the DGA. An **augmentation** ε of $\mathcal{A}(\Lambda)$ is a differential algebra homomorphism $\varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{Z}_2, 0)$; a **graded augmentation** is an augmentation so that ε is supported on elements of degree 0. Observe that, for any Legendrian Λ, there are only a finite number of augmentations. Given a graded augmentation ε, one can linearize $(\mathcal{A}(\Lambda), \partial)$ to a finite dimensional differential graded complex $(A(\Lambda), \partial^\varepsilon)$ and obtain **linearized contact homology**, denoted $LCH_*(\Lambda, \varepsilon; \mathbb{Z}/2)$, and its dual **linearized contact cohomology**, $LCH^*(\Lambda, \varepsilon; \mathbb{Z}/2)$. The set of all linearized (co)homology groups with respect to all possible graded augmentations is an invariant of Λ. If the augmenation is ungraded, one can still examine the rank of the non-graded linearized (co)homology, $\dim LCH(\Lambda, \varepsilon; \mathbb{Z}/2)$, and obtain as an invariant of Λ the set of ranks of this total linearized (co)homology for all possible augmentations. Examining ungraded linearized (co)homology is not an effective invariant: of the many examples of Legendrians knots in the Legendrian knot atlas of Chongchitmate and Ng, [11], that have the same classical invariants yet can be distinguished through graded Linearized homology, none of these can be distinguished by examining ungraded homology. However, ungraded (co)homology will be useful in arguments below.

Ekholm, [15], has shown that an exact Lagrangian filling, F, of a Legendrian submanifold $\Lambda \subset \mathbb{R}^3$ induces an augmentation ε_F of $(\mathcal{A}(\Lambda), \partial)$. When this filling has Maslov class 0, the augmentation will be graded.

The following result of Seidel will play a central role in showing obstructions to exact, non-orientable Lagrangian endocobordisms. A proof of this result was sketched by Ekholm in [16] and given in detail in Dimitroglou-Rizell, [13]; a parallel result using generating family homology is given in [38].

Theorem 4.1 (Seidel Isomorphism, [16], [13], [17])**.** *Let $\Lambda \subset \mathbb{R}^3$ Legendrian submanifold with an exact Lagrangian filling F; let ε_F denote the augmentation induced by the filling. Then*

$$\dim H(F; \mathbb{Z}/2) = \dim LCH(\Lambda, \varepsilon_F; \mathbb{Z}/2).$$

If the filling F of the n-dimensional Legendrian has Maslov class 0, then a graded version of the above equality holds:

$$\dim H_{n-*}(F; \mathbb{Z}/2) = \dim LCH^*(\Lambda, \varepsilon_F; \mathbb{Z}/2).$$

The ungraded version of the Seidel Isomorphism will be used to prove that any Legendrian Λ that is exactly fillable does not have an exact, non-orientable Lagrangian endocobordism:

Proof of Theorem 1.2. For a contradiction, suppose that there is a Legendrian knot Λ that has an exact Lagrangian filling and an exact non-orientable Lagrangian endocobordism. Then by stacking the endocobordisms, Lemma 2.2, it follows that Λ has an infinite number of topologically distinct exact, non-orientable Lagrangian fillings. Each of these exact Lagrangian fillings induces an augmentation. Since there are only a finite number of possible augmentations, there must exist two topologically distinct fillings that induce the same augmentation. However, this gives a contradiction to the Seidel Isomorphism, Theorem 4.1. $\square$

Theorem 1.2 implies that on the set of Legendrian knots in $\mathbb{R}^3$ that are exactly fillable, orientably or not, the relation defined by exact, non-orientable Lagrangian cobordism is anti-reflexive. Thus, by stacking, Lemma 2.2, we immediately also see:

Corollary 4.2. *On the set of Legendrian knots in $\mathbb{R}^3$ that are exactly fillable, orientably or not, the relation $\sim$ defined by exact, non-orientable Lagrangian cobordism is anti-symmetric: $\Lambda_1 \sim \Lambda_2 \implies \Lambda_2 \not\sim \Lambda_1$.*

We now apply Theorem 1.2 to give examples of Legendrians that do not have exact, non-orientable Lagrangian endocobordisms. Hayden and Sabloff, [29], showed that every positive knot type has a Legendrian representative that has an exact, orientable Lagrangian filling. In addition, Lipman, Reinoso, and Sabloff have shown that every 2-bridge knot and every $+$-adequate knot has a Legendrian representative with an exact filling, [31]. Combining this with Theorem 1.2, immediately gives:

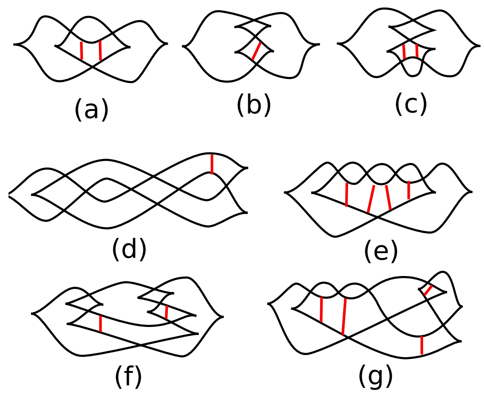

FIGURE 6. Examples of Legendrians that do not have exact, non-orientable Lagrangian endocobordisms: maximal tb representatives of (a) $m(3_1) = T(3,2) = K_{-2}$, (b) $3_1 = T(-3,2) = K_1$, (c) $4_1 = K_2 = K_{-3}$, (d) $5_1 = T(-5,2)$, (e) $m(5_1) = T(5,2)$, (f) 6_2, and (g) $m(6_2)$. The red lines indicate points for surgeries.

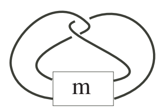

FIGURE 7. The smooth twist knot K_m; the box contains m right-handed half twists if $m \geq 0$, and $|m|$ left-handed twists if $m < 0$. Notice that K_0 and K_{-1} are unknots.

Corollary 4.3 ([29], [31])**.** *Each positive knot, 2-bridge knot, and +-adequate knot has a Legendrian representative that does not have an exact, non-orientable Langrangian endocobordism.*

Many maximal tb representatives of low crossing have fillings, orientable or not. Figure 6 illustrates some Legendrians that can be verified to have exact, Lagrangian fillings: see Remark 2.3. Many of the examples in Figure 6 are Legendrian representatives of twist or torus knots. Using Theorem 1.2 together with classification results of Etnyre and Honda, [23], and Etnyre, Ng, and Vértesi, [24], we show that *all* maximal tb represenatives of twist knots, positive torus knots, and negative torus knots of the form $T(-p, 2k)$, $p > 2k > 0$, do not have exact, non-orientable Lagrangian endocobordisms:

Proof of Corollary 1.3. By Theorem 1.2, to show the non-existence of an exact, non-orientable Lagrangian endocobordism, it suffices to show the existence of an exact Lagrangian filling.

First consider the case where Λ is a maximal tb representative of a twist knot, whose form is shown in Figure 7. Etnyre, Ng, and Vértesi, have classified all Legendrian twist knots, [24]: every maximal tb Legendrian

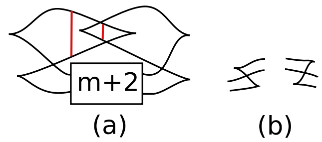

FIGURE 8. Any maximal tb Legendrian representative of a negative twist knot, K_m with $m \leq -2$, is Legendrian isotopic to one of the form in (a) where the box contains $|m+2|$ half twists, each of form S as shown in or of form Z as shown in (b). Two surgeries produces a max tb Legendrian unknot.

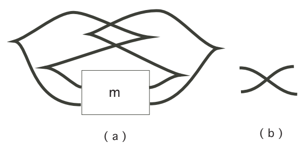

FIGURE 9. Any maximal tb Legendrian representative of a positive twist knot, K_m with $m \geq 1$, is Legendrian isotopic to one of the form in (a) where the box contains m half twists, each of form X as shown in (b).

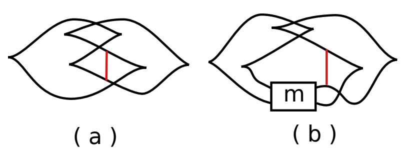

FIGURE 10. An inductive argument shows that every max tb representative of a positive twist knot has an exact Lagrangian filling.

representative of K_m, for $m \leq -2$, is Legendrian isotopic to one of the form in Figure 8, and every maximal tb Legendrian representative of K_m, for $m \geq 1$, is Legendrian isotopic to one of the form in Figure 9. For a max tb representative of a negative twist knot, Figure 8 illustrates the two surgeries that show the existence of an exact Lagrangian filling. For a max tb Legendrian representative of a positive twist knot, the existence of an exact filling can be shown by an induction argument: Figure 10 (a), indicates surgery point when $m = 1$; for all $m \geq 1$, a maximal tb representative of K_{m+1} can be reduced to a maximal tb representative of K_m by one surgery as indicated in Figure 10 (b).

Next consider maximal tb Legendrian representatives of a torus knot, a knot that can be smoothly isotoped so that it lies on the surface of an unknotted torus in $\mathbb{R}^3$. Every torus knot can be specified by a pair (p, q) of coprime integers: we will use the convention that the (p, q)-torus knot, $T(p, q)$, winds p times around a meridonal curve of the torus and q times

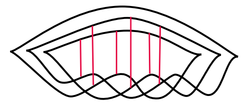

FIGURE 11. Surgeries that result in an exact filling of the maximal *tb* representative of the positive torus knot $T(5,3)$.

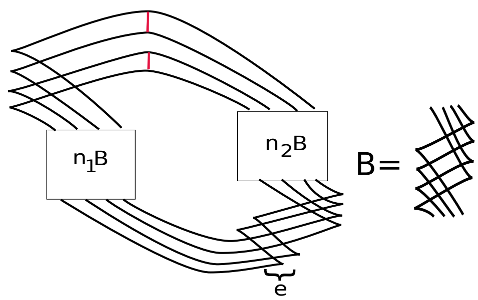

FIGURE 12. The general form of a maximal *tb* representative of a negative torus knot $T(-p,2k)$, with $p > 2k > 0$, with $k = 2$ and $|p| = (1+n_1+n_2)(2k)+e$; k surgeries produce a trivial Legendrian link of maximal *tb* unknots.

in the longitudinal direction. In fact, $T(p,q)$ is equivalent to $T(q,p)$ and to $T(-p,-q)$. We will always assume that $|p| > q \geq 2$, since we are interested in non-trivial torus knots.

Etnyre and Honda, [23], showed there is a unique maximal *tb* representative of a positive torus knot, $T(p,q)$ with $p > 0$. The surgeries used in [5, Theorem 4.2] show that each maximal representative is exactly fillable. Figure 11 illustrates the orientable surgeries for $(5,3)$-torus knot; in this sequence of surgeries, one begins with surgeries on the innermost strands, and then performs a Legendrian isotopy so that it is possible to do a surgery on the next set of innermost strands.

Lastly consider the case where Λ is topologically a negative torus knot, $T(-p,2k)$ with $p > 2k > 0$. In this case, Etnyre and Honda have shown that the number of different maximal *tb* Legendrian representations depends on the divisibility of p by $2k$: if $|p| = m(2k)+e$, $0 < e < 2k$, there are m non-oriented Legendrian representatives of $T(-p,2k)$ with maximal *tb*. These different representatives with maximal *tb* are obtained by writing $m = 1+n_1+n_2$, where $n_1, n_2 \geq 0$, and then $\Lambda_{(n_1,n_2)}$ is constructed using the form shown in Figure 12 with n_1 and n_2 copies of the tangle B inserted as indicated; this figure also shows k surgeries that guarantee the existence of an exact Lagrangian filling. □

Some comments on obstructions to exact fillings are discussed in Section 6.

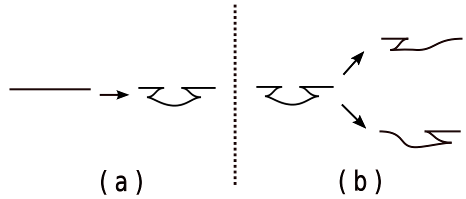

FIGURE 13. Via isotopy and surgeries, at least one of which is non-orientable, it is possible to construct exact non-orientable Lagrangian cobordisms between (a) $\Lambda_+ = \Lambda$ and $S_-S_+(\Lambda)$, (b) $\Lambda_+ = S_-S_+(\Lambda)$ and $\Lambda_- = S_+(\Lambda)$ or $\Lambda_- = S_-(\Lambda)$.

5. Constructions of Exact, Non-orientable Lagrangian Cobordisms

In this section, we will construct an exact, non-orientable Lagrangian endocobordisms of crosscap genus 4 for any stabilized Legendrian knot, and a non-orientable Lagrangian cobordism between any two stabilized Legendrian knots. All these exact Lagrangian cobordisms are constructed through isotopy and surgery, see Remark 2.3.

Central to these constructions will be the following lemma, which says that with respect to either orientation on Λ_+ one can always introduce a pair of oppositely oriented zig-zags, and if one has a pair of oppositely oriented zig-zags in Λ_+, then one can remove either element of this pair; see Figure 13.

Lemma 5.1. *Let Λ be any oriented Legendrian knot. Then there exists an exact, non-orientable Lagrangian cobordism:*

(1) *of crosscap genus 2 between $\Lambda_+ = \Lambda$ and $\Lambda_- = S_-S_+(\Lambda)$;*
(2) *of crosscap genus 1 between $\Lambda_+ = S_-S_+(\Lambda)$ and $\Lambda_- = S_+(\Lambda)$ or $\Lambda_- = S_-(\Lambda)$.*

Remark 5.1. With non-orientable cobordisms, given an orientation on Λ_+, there is no canonical orientations for Λ_-. In Lemma 5.1, an orientation on Λ_+ is chosen so that there are well-defined $S_-(\Lambda)$ and $S_+(\Lambda)$, but the statement implies that Λ_- can be $S_-S_+(\Lambda)$, $S_+(\Lambda)$, or $S_-(\Lambda)$ with either orientation.

Proof. The strategy will be to construct the desired exact, non-orientable Lagrangian cobordism via Legendrian isotopy and surgeries that are performed on a portion of a strand. Figure 14 illustrates the isotopy and surgeries, the second of which is non-orientable, that implies the existence of a crosscap genus 2 Lagrangian cobordism between $\Lambda_+ = \Lambda$ and $\Lambda_- = S_-S_+(\Lambda)$. Figure 15 illustrates the isotopy and surgery that implies the existence of a crosscap genus 1 Lagrangian cobordism between $\Lambda_+ = S_-S_+(\Lambda)$ and

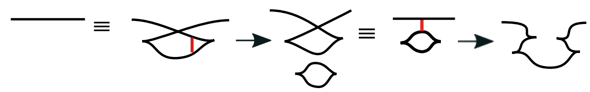

FIGURE 14. By applying an orientable and a non-orientable surgery, any strand can have a pair of oppositely oriented zig-zags introduced.

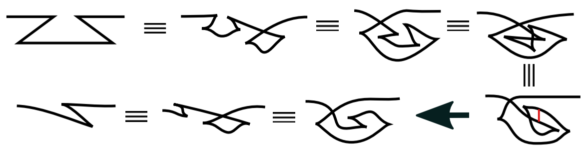

FIGURE 15. In the presence of oppositely oriented zig-zags, via one non-orientable surgery, one of the zig-zags can be removed.

$\Lambda_- = S_+(\Lambda)$, when the original strand is oriented from right to left, or to $\Lambda_- = S_-(\Lambda)$, when the original strand is oriented from left to right. $\square$

5.1. **Exact, Non-Orientable Lagrangian Endocobordisms.** In Theorem 1.2, it was shown that Legendrians that are exactly fillable do not have exact, non-orientable Lagrangian endocobordisms. However exact, non-orientable Lagrangian endocobordisms do exist for stabilized knots:

Proof of Theorem 1.4. First consider the case where Λ is the negative stabilization of a Legendrian: $\Lambda = S_-(\widehat{\Lambda})$. Then by applying Lemma 5.1, there exists an exact, non-orientable Lagrangian cobordism:

(1) of crosscap genus 2 between Λ and $S_-S_+(\Lambda)$;
(2) of crosscap genus 1 between $S_-S_+(\Lambda)$ and $S_+(\Lambda)$;
(3) of crosscap genus 1 between $S_+(\Lambda) = S_+(S_-(\widehat{\Lambda}))$ and $S_-(\widehat{\Lambda}) = \Lambda$.

Stacking these cobordisms results in an exact, non-orientable Lagrangian endocobordism of crosscap genus 4. Additional stacking results in arbitrary multiples of crosscap genus 4.

An analogous argument proves the case where Λ is the positive stabilization of a Legendrian: $\Lambda = S_+(\widehat{\Lambda})$. $\square$

5.2. **Exact, Non-Orientable Lagrangian Cobordisms between Stabilized Legendrians.** Given that every stabilized Legedendrian knot has a non-orientable Lagrangian endocobordism, a natural question is: What Legendrian knots can appear as a "slice" of such an endocobordism? In this section, we show that *any* stabilized Legendrian knot can appear as such a slice.

Theorem 5.2. *For smooth knot types K, K', let Λ be any Legendrian representative of K and let Λ' be a stabilized Legendrian representative of K'.*

Then there exists an exact, non-orientable Lagrangian cobordism between $\Lambda_+ = \Lambda$ *and* $\Lambda_- = \Lambda'$.

Before moving to the proof of Theorem 5.2, we show that non-orientable Lagrangian cobordisms define an equivalence relation on the set of stabilized Legendrian knots:

Proof of Theorem 1.6. Let $\mathcal{L}^s$ denote the set of all stabilized Legendrian knots of any smooth knot type. Define the relation $\sim$ on $\mathcal{L}^s$ by $\Lambda_1 \sim \Lambda_2$ if there exists an exact, non-orientable Lagrangian cobordism from $\Lambda_+ = \Lambda_1$ to $\Lambda_- = \Lambda_2$. Reflexivity of $\sim$ follows from Theorem 1.4. Symmetry of $\sim$ follows from Theorem 5.2. Transitivity of $\sim$ follows from Lemma 2.2. Thus $\sim$ defines an equivalence relation. Moreover, by Theorem 5.2, we see that with respect to this equivalence relation, there is only one equivalence class. □

To prove Theorem 5.2, it will be useful to first show that there is an exact, non-oriented Lagrangian cobordism between any two stabilized Legendrians of a fixed knot type:

Proposition 5.3. *Let K be any smooth knot type, and let Λ, Λ' be Legendrian representatives of K where Λ' is stabilized. Then there exists an exact, non-orientable Lagrangian cobordism between $\Lambda_+ = \Lambda$ and $\Lambda_- = \Lambda'$.*

Proof. Fix a smooth knot type K, and let Λ_1, Λ_2 be Legendrian representatives where Λ_2 is stabilized. By results of Fuchs and Tabachnikov, [26], we know that there exists r_1, ℓ_1, r_2, ℓ_2 so that $S_-^{\ell_1} S_+^{r_1}(\Lambda_1) = S_-^{\ell_2} S_+^{r_2}(\Lambda_2)$. By applying additional positive stabilizations, if needed, we can assume $r_1 > \ell_1$.

Consider the case where Λ_2 is the negative stabilization of some Legendrian: $\Lambda_2 = S_-(\hat{\Lambda}_2)$. By applications of Lemma 5.1, there exists an exact, non-orientable Lagrangian cobordism between:

(1) Λ_1 and $S_-^{r_1} S_+^{r_1}(\Lambda_1)$;
(2) $S_-^{r_1} S_+^{r_1}(\Lambda_1)$ and $S_-^{\ell_1} S_+^{r_1}(\Lambda_1)$, and thus between $S_-^{r_1} S_+^{r_1}(\Lambda_1)$ and $S_-^{\ell_2} S_+^{r_2}(\Lambda_2)$;
(3) $S_-^{\ell_2} S_+^{r_2}(\Lambda_2)$ and $S_+^{r_2}(\Lambda_2)$;
(4) $S_+^{r_2}(\Lambda_2) = S_+^{r_2}(S_-(\hat{\Lambda}_2))$ and $S_-(\hat{\Lambda}_2) = \Lambda_2$.

By stacking these cobordisms (Lemma 2.2), we have our desired exact, non-orientable Lagrangian cobordism between Λ_1 and Λ_2. An analogous argument proves the case where Λ_2 is the positive stabilization of some Legendrian. □

Proof of Theorem 5.2. The strategy here is to first show that one can construct an exact, non-orientable Lagrangian cobordism between Λ and a stabilized Legendrian unknot Λ_0. Similarly, it is possible to construct an exact, non-orientable Lagrangian cobordism between Λ' and a stabilized Legendrian unknot Λ_0'; we will show it is possible to "reverse" this sequence of surgeries and construct an exact, non-orientable Lagrangian cobordism between Λ_0' and $\widetilde{\Lambda}'$, which is a stabilization of Λ'. By Proposition 5.3, there

exists an exact, non-orientable Lagrangian cobordism between Λ_0 and Λ'_0 and between $\widetilde{\Lambda}'$ and Λ'. Thus by stacking, we will have the desired exact, non-orientable Lagrangian cobordism between Λ and Λ'.

We first show how it is possible to construct an exact, non-orientable Lagrangian cobordism from Λ to a Legendrian unknot; cf., [5]. Let Λ be an arbitrary stabilized Legendrian knot. We can assume that Λ has at least one positive crossing by, if necessary, applying a Legendrian Reidemeister 1 move. As shown in Figure 16, performing an orientable or non-orientable surgery near a crossing produces a crossing that can be removed through Legendrian Reidmeister moves. Perform such a surgery on every crossing in Λ until you have obtained k disjoint stabilized Legendrian unknots; since Λ has at least one positive crossing, we have performed at least one non-orientable surgery. Align the k Legendrian unknots vertically and perform orientable or non-orientable surgeries so that we obtain a single stabilized Legendrian unknot Λ_0. In this way, we have constructed an exact, non-orientable Lagrangian cobordism between Λ and Λ_0.

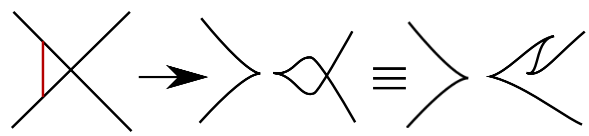

FIGURE 16. For any Legendrian knot Λ, perform a surgery near each crossing in order to get a disjoint set of Legendrian unknots.

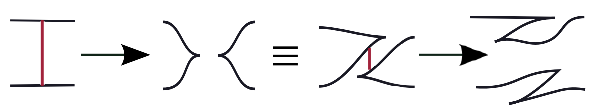

FIGURE 17. Surgeries used to convert to a link of Legendrian unknots can be "undone", at the cost of additional stabilizations.

A similar procedure can be used to construct a sequence of surgeries from Λ' to another Legnedrian unknot Λ'_0; now we show it is possible to "reverse" this procedure and construct a sequence of surgeries from Λ'_0 to $\widetilde{\Lambda}'$, a Legendrian obtained by applying stabilizations to Λ'. Figure 17 illustrates how every surgery that was used to get to a Legendrian unknot can be undone at the cost of adding additional zig-zags into the original strands. Figure 18 illustrates this procedure in a particular example.

As outlined at the beginning at the proof, these constructions prove the existence of an exact Lagrangian cobordism from $\Lambda_+ = \Lambda$ to $\Lambda_- = \Lambda'$. $\square$

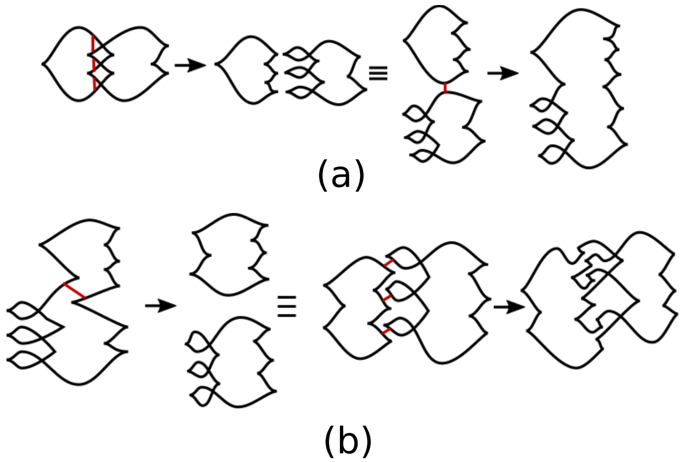

FIGURE 18. (a) Surgeries that give rise to an exact non-orientable Lagrangian cobordism from the max *tb* version of 3_1 to a stabilized unknot. (b) Surgeries that give rise to an exact non-orientable Lagrangian cobordism from the stabilized unknot to a stabilized representative of 3_1.

6. Additional Questions

We end with a brief discussion of some additional questions.

From results above, we know that exactly fillable Legendrian knots do not admit exact, non-orientable Lagrangian endocobordisms while stabilized Legendrian knots do. There are examples of Legendrian knots that are neither exactly fillable nor stabilized. As mentioned above, Ekholm, [15], has shown that if Λ is exactly fillable, then there exists an ungraded augmention of $\mathcal{A}(\Lambda)$. By work of Sabloff, [37], and independently, Fuchs and Ishkhanov, [25], we then know that there exists an ungraded ruling of Λ. Then it follows by work of Rutherford, [36], that the Kauffman bound on the maximal *tb* value for all Legendrian representatives of the smooth knot type of Λ is sharp. Thus, if the Kauffman bound is not sharp for the smooth knot type K, then no Legendrian representative of K is exactly fillable.

Question 6.1. *If Λ is a maximal tb representative of a knot type K for which the upper bound on tb for all Legendrian representatives given by the Kauffman polynomial is not sharp, does Λ have an exact, non-orientable Lagrangian endocobordism?*

The Legendrian representative of $m(8_{19})$ mentioned in Question 1.5 satisfies the hypothesis in Question 6.1. A list of some additional smooth knot types where the Kauffman bound is not sharp can be found in [34, Section 4].

There are also examples of Legendrians with non-maximal *tb* that are not stabilized. For example, $m(10_{161})$ is a knot type where the unique maximal *tb* representative has a filling. However, there are Legendrian representatives

with non-maximal tb that do not arise as a stabilization. As shown in [40, Figure 1], this Legendrian does have an ungraded ruling.

Question 6.2. *Does the non-stabilized, non-maximal tb Legendrian representative of $m(10_{161})$ have an exact, non-orientable Lagrangian endocobordism?*

Additional examples of non-stabilized and non-maximal tb representatives can be found in the Legendrian knot atlas of Chongchitmate and Ng, [11].

There are additional questions that arise from the constructions of fillings. For example, it is known by results of Chantraine, [8], that orientable fillings realize the smooth 4-ball genus. In Figure 6, examples are given of non-orientable Lagrangian fillings of maximal tb representatives of 6_2 and $m(6_2)$ of crosscap genus 2 and 4: the smooth 4-dimensional crosscap number of both 6_2 and $m(6_2)$ is 1.

Question 6.3. *Does there exist a non-orientable Lagrangian filling of these Legendrian representatives of 6_2 and $m(6_2)$ of crosscap genus 1?*

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